Lecture 12
Independence and Physical Probability

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Introduction

Two principles

In the theory of pp, a central question ought to be when pp's exist. In Lecture 4 I advocated two principles relevant to this:

1. Pp's are for outcome types relative to experiment types. Propositions don't have pp's.
2. Pp exists for experiment type X only if repeated performances of X are independent.

Von Mises accepted both conditions. Most contemporary pp theorists accept neither and have nothing to say about when pp's exist. E.g., David Lewis:

> *I think of chance* [pp] *as attaching in the first instance to propositions . . . It is only caution, not any definite reason to think otherwise, that stops me from thinking that chance of truth applies to any proposition whatever.* (1986 pp. 90–91)

Levi (1980, 1990) accepts (1) but denies (2). I don't know any non-frequency theorist that accepts (2), other than me.

What I'll do today

- State the independence principle more precisely than I did before.
- Present and refute Levi's argument against the independence principle.
- State a *new* principle which is even more useful in determining when pp's exist.

In this lecture, I focus on the explicandum, pp, rather than an explicatum. That allows me to use intuitive numbers in examples, without having to prove that they hold for the explicatum.

Independence

Conditional pp

- So far, I've talked about pp_X as having a single argument; it satisfies the laws of *unconditional* probability.
- There is also *conditional* pp; it satisfies:

$$pp_X(O_2|O_1) = \frac{pp_X(O_2.O_1)}{pp_X(O_1)}, \text{ provided } pp_X(O_1) > 0.$$

Example

X = tossing a symmetric die, O_1 = the die comes up a number greater than 3, O_2 = the die comes up an even number.

$$\begin{aligned} pp_X(O_2) &= 1/2; \\ pp_X(O_2|O_1) &= \frac{pp_X(4 \text{ or } 6)}{pp_X(4,\ 5, \text{ or } 6)} = \frac{2}{3}. \end{aligned}$$

Definition

$O_1, \ldots, O_n$ are *independent in pp_X* iff, for any distinct $i_1, \ldots, i_m \in \{1, \ldots, n\}$,

$$pp_X(O_{i_1} \ldots O_{i_m}) = pp_X(O_{i_1}) \ldots pp_X(O_{i_m}).$$

Theorems

1. O_1 and O_2 are independent in pp_X iff
$$pp_X(O_1.O_2) = pp_X(O_1)\, pp_X(O_2).$$
2. If $pp_X(O_1) > 0$ then O_1 and O_2 are independent in pp_X iff
$$pp_X(O_2|O_1) = pp_X(O_2).$$

I omit "in pp_X" when it is obvious what is meant.

Examples

1. X = drawing a card from a normal 52-card deck, O_1 = the card is an ace, O_2 = the card is a heart.

$$pp_X(O_2|O_1) = \frac{1}{4} = pp_X(O_2).$$

   So O_1 and O_2 are independent.

2. X = tossing a symmetric die, O_1 = the die comes up a number greater than 3, O_2 = the die comes up an even number. We found:

$$pp_X(O_2|O_1) = \frac{2}{3} > \frac{1}{2} = pp_X(O_2).$$

   So O_1 and O_2 are dependent.

Experiments with repeated actions

Repeated performances of an action can be treated as one experiment. Outcomes of different parts of such an experiment may be dependent or independent.

Examples

1. X = drawing two cards from a normal deck of 52 cards, without replacement; O_i = the ith card is an ace. Here O_1 and O_2 are dependent.

$$
\begin{aligned}
pp_X(O_2|O_1) &= 3/51; \\
pp_X(O_2) &= pp_X(O_2|O_1)pp_X(O_1) + pp_X(O_2|{\sim}O_1)pp_X({\sim}O_1) \\
&= (3/51)(1/13) + (4/51)(12/13) = 1/13.
\end{aligned}
$$

2. The same except the first card is replaced and the deck shuffled before the second card is drawn. Here O_1 and O_2 are independent:

$$pp_X(O_2|O_1) = 1/13 = pp_X(O_2).$$

Independence principle

Notation:

- O_i is a possible outcome of X;
- X^n is the experiment of performing X n times;
- $O_i^{(k)}$ is the outcome of X^n which consists in getting O_i on the kth performance of X.

Independence principle (IN)

If pp_X exists then, for all positive integers n, pp_{X^n} exists and

$$pp_{X^n}(O_1^{(1)} \dots O_n^{(n)}) = pp_X(O_1) \dots pp_X(O_n).$$

Equivalent formulation

The identity in IN doesn't fit the definition of independent outcomes because it concerns two different probability functions. But the following are equivalent:

1. For all positive integers n,
$$pp_{X^n}(O_1^{(1)} \dots O_n^{(n)}) = pp_X(O_1) \dots pp_X(O_n).$$
2. For all positive integers n,
   (a) $pp_{X^n}(O_i^{(i)}) = pp_X(O_i)$; and
   (b) $O_1^{(1)}, \dots, O_n^{(n)}$ are independent in pp_{X^n}.

Proof that (1) implies (2):

Let all $O_1, \dots, O_n$ except O_i be necessary outcomes; then (1) gives (2a). Substituting (2a) in (1) gives (2b).

Proof that (2) implies (1):

(2b) says $pp_{X^n}(O_1^{(1)} \dots O_n^{(n)}) = pp_{X^n}(O_1^{(1)}) \dots pp_{X^n}(O_n^{(n)})$. Substituting (2a) in this gives (1).

Examples

1. $X =$ tossing a coin once. Here pp_X exists, so does pp_{X^n}, and for any $O_1, \ldots, O_n$ that are possible outcomes of X,

$$pp_{X^n}(O_1^{(1)} \ldots O_n^{(n)}) = pp_X(O_1) \ldots pp_X(O_n).$$

2. $X =$ shuffling a normal deck of 52 cards and then drawing two cards without replacement. Here pp_X exists, so does pp_{X^n}, and for any $O_1, \ldots, O_n$ that are possible outcomes of X,

$$pp_{X^n}(O_1^{(1)} \ldots O_n^{(n)}) = pp_X(O_1) \ldots pp_X(O_n).$$

3. $Y =$ one of the drawings in the previous example. Intuitively, pp_Y doesn't exist because it matters what cards are left in the deck.

In all these cases, pp satisfies IN.

Questions

1. What does it mean for $O_1, \ldots, O_n$ to be independent in pp_X?
2. Prove that if $pp_X(O_1) > 0$ then O_1 and O_2 are independent in pp_X iff $pp_X(O_2|O_1) = pp_X(O_2)$.
3. State the independence principle IN.
4. Prove that (i) and (ii) are equivalent:
   (i) For all positive integers n,
   $$pp_{X^n}(O_1^{(1)} \ldots O_n^{(n)}) = pp_X(O_1) \ldots pp_X(O_n).$$
   (ii) For all positive integers n,
   (a) $pp_{X^n}(O_i^{(i)}) = pp_X(O_i)$; and
   (b) $O_1^{(1)}, \ldots, O_n^{(n)}$ are independent in pp_{X^n}.
5. An urn contains five white balls and five black balls. Let $X =$ drawing two balls from the urn without replacement, $O_i =$ the ith ball is white. Are O_1 and O_2 independent in pp_X? Is this a counterexample to IN? Justify your answers.

Levi's rejection of IN

Changing pp's (1980 p. 272)

Levi considers a postulate equivalent to IN and argues that it doesn't hold in general.

> [A person] *might believe that coin* a *is not very durable so that each toss alters the chance of heads on the next toss and that how it alters the chance is a function of the result of the previous tosses.* [The person] *might believe that coin* a*, which has never been tossed, has a .5 chance of landing heads on a toss as long as it remains untossed. Yet, he might not believe that the chance of r heads on n tosses is* $\binom{n}{r}(.5)^n$.

The latter formula follows from IN and $pp_X(\text{heads}) = .5$.

Levi is inconsistent

Levi seems to be saying that the pp of experiment type X giving outcome type O can be different for different tokens of X. He explicitly asserts that elsewhere:

> *When a trial of some kind can be repeated, the chances of response may change from trial to trial.* (1990 p. 128)

This is inconsistent with the view, which Levi accepts, that pp is a function of the experiment and outcome types.

How to deal with Levi's coin

- We may take X to be starting with the coin symmetric and tossing it n times. This is an experiment with repeated actions and the outcomes of those various actions can be dependent. What IN requires is only that different performances of X be independent, which Levi doesn't deny.
- We may take X to be tossing the coin when it is in such-and-such a state. In that case, X is different for different tosses of the coin. Again, there is no violation of IN.
- Levi seems to take X to be tossing the coin, without specifying the state that it is in. I think it is plausible that pp doesn't exist for this experiment type. (Levi's alternative, that it exists but changes from token to token, is inconsistent.)

The coin example doesn't introduce anything new; the same issues arise in the standard examples of drawing cards from a pack, or balls from an urn, without replacement.

Unrepeatable experiments (Levi 1980 p. 273)

It is also high time to question the old dogma that kinds of trials for which chances are definable should be repeatable. Suppose bottle a is disposed to break into pieces when dropped. The kind of trial (dropping the bottle) is not repeatable on the setup a. Yet the bottle may be said to have the disposition. We can also say that the bottle a has a chance p of breaking into 10 pieces when dropped even though the trial is not repeatable. Illustrations of the nonrepeatability of kinds of trials on the same setup abound in quantum mechanics. The phenomenon, however, is not restricted to that domain.

Repeatability not required

- Levi's examples are not compelling.
  - It isn't obvious that $pp_X(O)$ exists when $X =$ dropping this particular bottle and $O =$ breaking into 10 pieces.
  - In quantum mechanics, the pp's that scientists refer to are for experiments on a certain *type* of system; they aren't for experiments on one token system.
- I would go further than Levi: pp_X can exist even when X can *never* be performed.

  E.g., let $X =$ tossing a symmetric coin a trillion times in a way that does not alter the coin at all. This can't be done even once but pp_X exists. E.g., if $O =$ the coin lands heads every time, $pp_X(O) = 2^{-2^{12}}$.
- This isn't a problem for IN. What IN says is that if pp_X exists then pp_{X^n} exists. Since pp_X can exist when X is not performable, the assertion that pp_{X^n} exists does not imply that X is repeatable.

Existence of pp

Levi's disjunctive example (Levi 1980 p. 264)

Suppose box a has two compartments. The left compartment contains 40 black balls and 60 white balls and the right compartment contains 40 red balls and 60 blue balls. A trial of kind S is selecting a ball at random from the left compartment and a trial of kind S' is selecting a ball at random from the right compartment … Chances are defined for both kinds of trials over their respective sample spaces [possible outcomes].

Consider trials of kind $S \vee S'$. There is indeed a sample space consisting of drawing a red ball, a blue ball, a black ball, and a white ball. However, there is no chance distribution over the sample space.

Levi's explanation (Levi 1980 pp. 264–265)

To see why no chance distribution is defined, consider that the sample space for trials of kind $S \vee S'$ is such that a result consisting of obtaining a [black] *or a* [white] *ball is equivalent to obtaining a result of conducting a trial of kind S ... Thus, conducting a trial of kind $S \vee S'$ would be conducting a trial of kind S with some definite chance or statistical probability.*

There is no a priori consideration precluding such chances; but there is no guarantee that such chances are defined either. In the example under consideration, we would normally deny that they are.

To be sure, if we were to conduct a trial consisting of tossing a coin with a known chance of heads and of tails and then conduct a trial of [kind] *S if the coin lands heads and a trial of kind S' if the coin lands tails, there would be a definite chance of conducting a trial of kind S (and of kind S') on such a trial. But the trial is not merely of kind $S \vee S'$ ... The trial is of kind T, where all trials of kind T are trials of kind $S \vee S'$, but not conversely.*

The explanation is shallow

I agree with Levi that $pp_{S \vee S'}$ doesn't exist and pp_T does. However, Levi's "explanation" of this is very shallow:

- It rests on the assertion that $pp_{S \vee S'}(S)$ doesn't exist, for which Levi has no explanation.
- It depends on S and S' having no possible outcomes in common, though the phenomenon is not restricted to that special case.

The following principle provides a deeper explanation.

Specification principle (SP)

If it is possible to perform X in a way that ensures it is also a performance of the more specific experiment type X', then pp_X exists only if $pp_{X'}$ exists and is identical to pp_X.

Example

- X = tossing a coin in the usual way.
- X' = tossing a coin in the usual way on a Monday.

Here pp_X exists, and it is possible to perform X in a way that ensures it is also a performance of X', so SP implies that $pp_{X'}$ exists and equals pp_X—which it does.

Corollary

If it is possible to perform X in a way that ensures it is also a performance of the more specific experiment type X_i, for $i = 1, 2$, and if $pp_{X_1} \neq pp_{X_2}$, then pp_X doesn't exist.

Application to Levi's example

- It is possible to perform $S \vee S'$ in a way that ensures S is performed; likewise for S'. Since $pp_S \neq pp_{S'}$, it follows that $pp_{S \vee S'}$ doesn't exist.
- A full description of T must say that the toss is done by a human, without any special apparatus that could precisely fix the position of the coin, the force applied to it, etc. Then it isn't possible to perform T in a way that ensures some more specific experiment T' is performed, where $pp_{T'} \neq pp_T$. Thus, the existence of pp_T is compatible with SP.

Questions

6. Levi argued against IN by saying:

   > [A person] *might believe that coin a is not very durable so that each toss alters the chance of heads on the next toss and that how it alters the chance is a function of the result of the previous tosses.*

   Describe three possible identifications of the experiment type in this example and, for each, say whether the example violates IN, and why.

7. Does IN imply that X must be repeatable in order for pp_X to exist? Justify your answer.

8. Levi (1990 p. 120) said: "The chance of coin *a* landing heads on a toss may be 0.5, but the chance of the coin landing heads on a toss by Morgenbesser may, at the same time, be 0.9." Is that correct? Justify your answer.

9. If pp_X and $pp_{X'}$ exist, does it follow that $pp_{X \vee X'}$ exists? Explain.

References

- Levi, Isaac. 1980. *The Enterprise of Knowledge*. MIT Press. Paperback edition, with corrections, 1983.
- Levi, Isaac. 1990. "Chance," *Philosophical Topics 18* 2 (Fall), 117–149.
- Lewis, David. 1986. *Philosophical Papers* vol. 2, Oxford University Press.